Contextuality without Incompatibility

John H. Selby,[1,*] David Schmid,[1,2,3] Elie Wolfe,[2] Ana Belén Sainz,[1] Ravi Kunjwal,[4] and Robert W. Spekkens[2]
[1]*International Centre for Theory of Quantum Technologies, University of Gdańsk, 80-309 Gdańsk, Poland*
[2]*Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario Canada N2L 2Y5*
[3]*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*
[4]*Centre for Quantum Information and Communication, Ecole polytechnique de Bruxelles, CP 165, Université libre de Bruxelles, 1050 Brussels, Belgium*

(Received 16 February 2022; revised 10 February 2023; accepted 5 May 2023; published 6 June 2023)

The existence of incompatible measurements is often believed to be a feature of quantum theory which signals its inconsistency with any classical worldview. To prove the failure of classicality in the sense of Kochen-Specker noncontextuality, one does indeed require sets of incompatible measurements. However, a more broadly applicable notion of classicality is the existence of a generalized-noncontextual ontological model. In particular, this notion can imply constraints on the representation of outcomes even within a single nonprojective measurement. We leverage this fact to demonstrate that measurement incompatibility is neither necessary nor sufficient for proofs of the failure of generalized noncontextuality. Furthermore, we show that every proof of the failure of generalized noncontextuality in a quantum prepare-measure scenario can be converted into a proof of the failure of generalized noncontextuality in a corresponding scenario with no incompatible measurements.

DOI: 10.1103/PhysRevLett.130.230201

Measurement incompatibility—the existence of measurements that cannot be implemented simultaneously—has conventionally been taken to be part of what is truly distinctive about quantum theory relative to its classical forebears. Here, we are concerned with whether this attitude is justified when the notion of classicality at play is whether or not a theory (or experiment) admits of an ontological model satisfying the principle of generalized noncontextuality [1,2]. It is already known that classical statistical theories with an epistemic restriction [3] (and subtheories of quantum theory which make the same predictions as these) can manifest incompatibility despite satisfying the principle of generalized noncontextuality. Hence, the mere fact that a theory exhibits measurement incompatibility is not sufficient to infer that it must fail to admit of a generalized-noncontextual ontological model.

In this Letter, we demonstrate that measurement incompatibility is also not *necessary* for demonstrating that quantum theory (or an experiment within quantum theory) is nonclassical in this sense. In summary, we have

$$\text{Incompatibility} \quad \begin{array}{c}[1,2]\\ \not\Rightarrow \\ \not\Leftarrow \\ \text{Thm. 1}\end{array} \quad \begin{array}{c}\text{Impossibility of a}\\ \text{generalized-noncontextual}\\ \text{ontological model.}\end{array}$$

In a companion paper, Ref. [4], we give more general arguments and applications of these results (see the conclusions for a summary).

The notion of a generalized-noncontextual ontological model, introduced in Ref. [1], was proposed to overcome some of the limitations of the Kochen-Specker notion of noncontextuality, and has since been further refined [2]. It can be understood as a special case of a methodological principle due to Leibniz, a version of the principle of the identity of indiscernibles [2,5]. Unlike the Kochen-Specker notion, generalized noncontextuality has implications not only for sharp measurements but for all procedures, including unsharp measurements, preparations, and transformations. It has been found to subsume many other notions of classicality [6–10] and to shed light on the resource of nonclassicality underlying the quantum advantages known to exist for many information processing tasks [11–23].

Recall that the notion of Kochen-Specker noncontextuality is defined only for projective measurements (i.e., those whose outcomes correspond to the eigenspaces of a Hermitian operator), and the context independence of the ontological representation of a measurement is understood as the lack of dependence on what other measurement is implemented simultaneously with it. It is well-known that the notion of Kochen-Specker noncontextuality only implies nontrivial constraints on the ontological representation if the set of measurements under consideration includes some incompatible ones.

By contrast, the notion of generalized noncontextuality provides a much broader scope of possibilities for assuming context independence: namely, any two procedures which are operationally equivalent [1] are assumed to have the same ontological representation. As it turns out, for unsharp measurements—which in quantum theory are associated with a positive operator-valued measure (POVM) that is not projector valued—there can be nontrivial operational equivalences between the outcomes of one and the same measurement, so that generalized noncontextuality has nontrivial implications for the ontological representation of even a *single* such measurement. For instance, consider the POVM $\{\frac{1}{2}\mathbb{1}, \frac{1}{2}\mathbb{1}\}$ where $\mathbb{1}$ is the identity operator. The two outcomes of this measurement are equally likely on all states, and hence its two effects are operationally equivalent. By generalized noncontextuality, then, they must be represented in the ontological model by the *same* conditional probability distribution [24].

It is the existence of such nontrivial constraints on the ontological representation of a single measurement that opens up the possibility of a proof of the failure of generalized noncontextuality without incompatibility. Note that the possibility is open *only if* the measurement is unsharp—any experiment involving a single projective measurement trivially admits of a generalized-noncontextual model. We demonstrate here that this possibility is in fact realized, and it is realized in the simplest operational scenario, namely, a prepare-measure experiment.

Our main result can be summarized as follows:

Theorem 1.—From every quantum proof of the failure of generalized noncontextuality in a prepare-measure scenario involving incompatible measurements one can construct a proof that does not require any incompatibility.

In the following, the term "noncontextual" will be used as a shorthand for "generalized noncontextual." We will also use the expression "proof of contextuality" as a shorthand for "proof of the impossibility of a generalized-noncontextual ontological model" (even though the lesson of such proofs may well be to reject the framework of ontological models while maintaining the Leibnizian methodological principle that underlies noncontextuality [2]).

Reference [25] makes a claim which is superficially contrary to ours, namely, that incompatibility is necessary and sufficient for proving generalized contextuality. The claim of *sufficiency* in Ref. [25] is only established relative to the strong assumption that the set of states under consideration includes *all* quantum states. This assumption is violated in any real experiment and also by many interesting subtheories of quantum theory [3,26,27]. The claim of the *necessity* of incompatibility in Ref. [25] is predicated on assuming noncontextuality for preparations alone rather than for preparations and measurements.

However, the motivation for assuming noncontextuality for preparations (namely, the Leibnizian methodological principle [2,5]) is also a motivation for assuming it for measurements, and consequently it is unnatural to restrict the scope of the assumption to preparations alone.

Conventional no-go theorems for generalized noncontextuality.—We now introduce the requisite preliminaries by describing a conventional no-go theorem for generalized noncontextuality in the usual setting of prepare-measure scenarios on a given quantum system $\mathcal{H}$. Such a scenario is depicted in Fig. 1(a). The scenario is characterized by a set of quantum states $\{\rho_s\}_s$, indexed by the preparation setting $s \in S$, and a set of POVMs, $\{\{E_{y|t}\}_y\}_t$, indexed by the measurement setting $t \in T$, with elements (termed *effects*) in a given measurement indexed by outcomes $y \in Y$. (Note that taking the set Y to be the same cardinality for all values $t \in T$ involves no loss of generality.) The observable statistics in this scenario are given by the Born rule, i.e.,

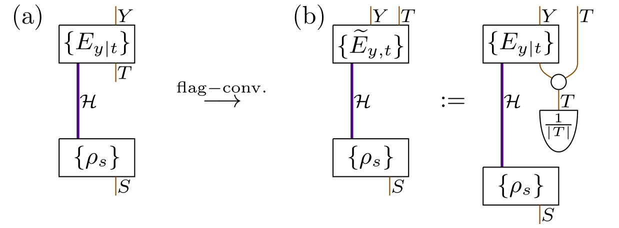

FIG. 1. (a) The original prepare-measure scenario. (b) The flag-convexified scenario, where the white dot represents the copying of T.

$$P^{(\mathrm{q})}(y|s,t) = \mathrm{tr}(E_{y|t}\rho_s). \tag{1}$$

The features of an operational scenario which drive any proof of contextuality are the set of operational equivalences that hold among the preparations, indexed by $a \in O_P$, and the set of operational equivalences that hold among the measurements, indexed by $b \in O_M$. These can be expressed [28] as linear constraints on the corresponding states and effects:

$$\sum_s \alpha_s^{(a)}\rho_s = 0, \qquad \sum_{y,t} \beta_{yt}^{(b)} E_{y|t} = 0, \tag{2}$$

for all $a \in O_P$ and $b \in O_M$ where $\alpha_s^{(a)}$ and $\beta_{yt}^{(b)}$ are real coefficients which specify the operational equivalence.

An ontological model of the prepare-measure scenario associates an ontic state space Λ with the system $\mathcal{H}$ and explains the operational statistics as arising from stochastic processes on Λ. That is, each quantum state ρ_s is represented as a probability distribution $P(\lambda|s)$ over the ontic states, $\lambda \in \Lambda$, of the system, and each measurement outcome, associated with quantum effect $E_{y|t}$, is represented by a conditional probability distribution $P(y|t\lambda)$ describing the probability of outcome y occurring given that the measurement was t and that the ontic state was λ.

An ontological model respects the assumption of generalized noncontextuality if the ontological representations of procedures respect the operational equivalences that hold among these. For the operational equivalences of Eq. (2), generalized noncontextuality implies that

$$\sum_s \alpha_s^{(a)} P(\lambda|s) = 0, \qquad \sum_{y,t} \beta_{yt}^{(b)} P(y|t,\lambda) = 0, \tag{3}$$

for all $\lambda \in \Lambda$, $a \in O_P$, and $b \in O_M$.

The correlations $P(y|s,t)$ that can be realized as $P(y|s,t) = \sum_\lambda P(y|t,\lambda)P(\lambda|s)$ for $P(y|t,\lambda)$ and $P(\lambda|s)$ satisfying Eq. (3), i.e., those that are *noncontextually realizable* for the given operational equivalences, form a polytope. The facets of this polytope are examples of *noncontextuality inequalities* [28]. One obtains a proof of contextuality whenever the observed quantum correlations $P^{(\mathrm{q})}(y|s,t)$ violate at least one of these facet-defining noncontextuality inequalities. In such cases, there is no noncontextual ontological model that can reproduce the quantum correlations.

There are many known examples of such proofs [11–23,29–34]; however, to the best of our knowledge, all previous proofs have involved a set of measurements that manifest some incompatibility.

Measurement incompatibility.—Compatibility for generic quantum measurements—both sharp and unsharp—is defined in terms of joint simulability [35]. For the case of discrete outcome spaces, it can be expressed as follows [36,37]: the measurements associated to a set of POVMs $\{\{E_{y|t}\}_y\}_t$ are said to be *compatible* if there exists a single measurement, described by a POVM $\{G_z\}_z$, and a stochastic postprocessing $P(y|t,z)$ such that

$$E_{y|t} = \sum_z P(y|t,z)G_z \tag{4}$$

for all y, t. In this case, each measurement $\{E_{y|t}\}_y$ can be simulated by first measuring $\{G_z\}_z$ and then implementing a postprocessing of the outcome statistics by $P(y|t,z)$ [38]. If such a measurement and postprocessing do not exist, then the set of measurements is said to be *incompatible*.

Note that a set consisting of a single measurement is trivially compatible.

Generalized noncontextuality no-go theorems without incompatibility.—We now construct a class of operational scenarios that allow a proof of contextuality without making use of any measurement incompatibility.

We do so by starting with a prepare-measure scenario of the type described above—with a set of preparations associated to states $\{\rho_s\}_s$, a set of measurements associated to POVMs $\{\{E_{y|t}\}_y\}_t$, and operational equivalences described by Eq. (2)—and we implement a modification on the measurement side. Specifically, the modified scenario involves a single measurement which is obtained from the set of measurements in the original scenario by a procedure we term "flag-convexification." One flag convexifies the set of measurements by randomly sampling a setting $t \in T$ according to some probability distribution $P(t)$, then implementing the measurement for that setting, $\{E_{y|t}\}_y$, and finally outputting both y and t, so that $(y,t) \in Y \times T$ constitutes the outcome of the new effective measurement. The terminology for the procedure stems from the fact that one is taking a convex mixture of the measurements in the original scenario, but one wherein the choice of measurement is not forgotten but rather flagged, i.e., copied and fed forward to be included in the output.

To make the argument as simple as possible, we consider uniform sampling, i.e., $P(t) = (1/|T|)$, where $|T|$ is the cardinality of the set T of possible settings. (In our companion paper [4], we show that any distribution with full support also works.) This simple version of flag convexification is depicted in Fig. 1. The single measurement in the flag-convexified scenario is associated to a POVM $\{\tilde{E}_{y,t}\}_{y,t}$ defined by

$$\tilde{E}_{y,t} \coloneqq \frac{1}{|T|} E_{y|t}. \tag{5}$$

It is straightforward to check that this is a valid POVM. For ease of bookkeeping, effects and conditional probability distributions which refer to the flag-convexified scenario will be denoted with a tilde.

Note that the correlations in the original scenario are naturally associated with a conditional probability distribution of the form $P^{(\mathrm{q})}(y|s,t) \coloneqq \mathrm{tr}(E_{y|t}\rho_s)$ where t is on the right-hand side of the conditional, while the correlations in the flag-convexified scenario are naturally associated with a conditional probability distribution of the form $\tilde{P}^{(\mathrm{q})}(y,t|s) \coloneqq \mathrm{tr}(\tilde{E}_{y,t}\rho_s)$ where t is on the left-hand side of the conditional. Using Eq. (5) and the linearity of the Born rule, these are related simply by

$$\tilde{P}^{(\mathrm{q})}(y,t|s) = \frac{1}{|T|} P^{(\mathrm{q})}(y|s,t). \tag{6}$$

Since the set of preparations in the flag-convexified scenario is identical to the set in the original scenario, it follows that the operational equivalences for the preparations are unchanged relative to Eq. (2),

$$\sum_s \alpha_s^{(a)} \rho_s = 0, \tag{7}$$

for all $a \in O_P$. Consider now the operational equivalences among the effects in the new scenario. Substituting Eq. (5) into Eq. (2), one finds that the operational equivalences in the flag-convexified scenario are given by

$$\sum_{y,t} \beta_{yt}^{(b)} \tilde{E}_{y,t} = 0 \tag{8}$$

for all $b \in O_M$. So the effects $\tilde{E}_{y,t}$ in the flag-convexified scenario satisfy linear constraints of exactly the same form as those satisfied by the effects $E_{y|t}$ from the original scenario.

An ontological representation of the flag-convexified scenario represents each state ρ_s by some probability distribution $\tilde{P}(\lambda|s)$ and each effect $\tilde{E}_{y,t}$ by some conditional probability distribution $\tilde{P}(y,t|\lambda)$. Given the operational equivalences of Eqs. (7) and (8), the assumption of noncontextuality implies that $\tilde{P}(\lambda|s)$ and $\tilde{P}(y,t|\lambda)$ must satisfy

$$\sum_s \alpha_s^{(a)} \tilde{P}(\lambda|s) = 0, \qquad \sum_{y,t} \beta_{yt}^{(b)} \tilde{P}(y,t|\lambda) = 0, \tag{9}$$

for all $\lambda \in \Lambda$, $a \in O_P$, and $b \in O_M$, which are seen to be of the same form as Eq. (3).

In the Supplemental Material [41], we prove that there exists a noncontextual ontological model for the original scenario if and only if there exists a noncontextual ontological model for the flag-convexified scenario. This fact implies that every no-go theorem for noncontextuality in a prepare-measure scenario involving incompatibility (of which there are many) can be transformed via flag convexification into a no-go theorem in a scenario that involves only a single measurement (and hence involves no incompatibility). This establishes Theorem 1.

An example not built from processings of incompatible sharp measurements.—The opportunity for proving generalized contextuality without incompatibility arises from the fact that there can be nontrivial operational equivalences between the outcomes of one and the same measurement if the latter is unsharp. It is natural to ask whether this opportunity *only* presents itself when the unsharp measurement is a flag convexification (or stochastic postprocessing) of a set of incompatible sharp measurements and when the operational equivalences in the former are implied by the conventional variety of operational equivalences in the latter. If it did, then one might dismiss proofs *without* incompatibility as simply proofs *with* incompatibility "in disguise." As we now demonstrate, the question is answered in the negative—there are genuinely novel types of proofs of contextuality for unsharp measurements.

Consider a prepare-measure scenario with five preparations, associated with the set of normalized rank-1 projectors $\{\rho_s = |\psi_s\rangle\langle\psi_s|\}_{s=0}^4$ where

$$|\psi_s\rangle \coloneqq \cos\left(\frac{\pi}{5}s\right)|0\rangle + \sin\left(\frac{\pi}{5}s\right)|1\rangle \tag{10}$$

and a single five-outcome measurement, associated with the POVM $\{E_y\}_{y=0}^4$ where the effects are defined as $E_y \coloneqq \frac{2}{5}\rho_y$ (i.e., subnormalized versions of the projectors onto the states) and thus sum to the identity.

These quantum states and effects can be represented as real-valued vectors in the space of Hermitian operators using the basis of Pauli operators. Because they all have zero component of the Y Pauli, they can be represented in the three-dimensional space spanned by I, Z, and X Paulis. We provide these representations in Figs. 2(a) and 2(b) respectively. For five vectors in a three-dimensional space, there are necessarily linear dependences amongst them, which can be captured by a pair of equations. For the five

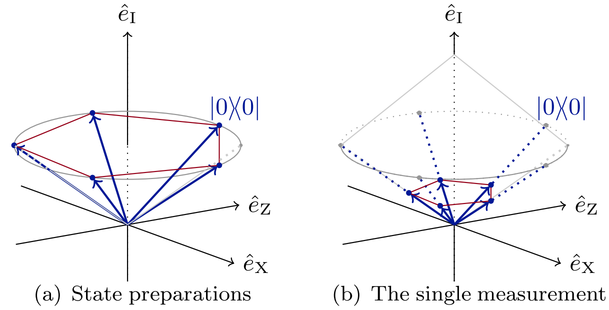

FIG. 2. An explicit example of contextuality without incompatibility. Bloch-vector representation of (a) the five quantum state preparations, and (b) the five outcomes of the single unsharp measurement. Since all operators in the example involve no component of the Pauli-Y matrix, we depict all operators by their three-dimensional vector representation (a_I, a_X, a_Z) defined by $A = a_I I + a_X X + a_Z Z$, where I, X, and Z are the other three Pauli operators.

states, these are interpreted as a pair of operational equivalences. They can be expressed as

$$\begin{aligned} \rho_0 - q\rho_1 + q\rho_2 - \rho_3 &= 0, \\ \rho_1 - q\rho_2 + q\rho_3 - \rho_4 &= 0, \end{aligned} \tag{11}$$

where q is the golden ratio, namely $q = 2\cos(\pi/5) = (1+\sqrt{5})/2$. The five effects satisfy the same operational equivalence relations [substituting E_i for ρ_i in Eq. (11)] because they are equal to the states up to a normalization factor.

By applying the linear programming techniques of Ref. [42] to this scenario, we can compute facet-defining noncontextuality inequalities. (Further details are provided in the Supplemental Material [41]).

Consider a conditional probability distribution $P(y|s)$ (whose elements we will abbreviate as $p_{y|s}$) and the following linear combination of these elements:

$$\mathcal{C} \coloneqq q(p_{1|0} + p_{1|2}) + (q-1)p_{2|0} + p_{0|2} - (q+1)p_{1|1}. \tag{12}$$

An inequality that defines a facet of the polytope of noncontextually realizable conditionals $P(y|s)$ is found to be

$$\mathcal{C} \geq 0. \tag{13}$$

Meanwhile, the quantumly realizable correlations for the scenario described above yield

$$\mathcal{C}^{(\mathrm{q})} = \frac{q^2 - 4}{10} \approx -0.138, \tag{14}$$

which violates the noncontextuality inequality of Eq. (13), thereby proving that the quantum scenario does not admit of a noncontextual model.

Finally, we demonstrate that the five-outcome POVM used in this proof cannot be understood as the flag convexification of a set of projector-valued measures (PVMs), nor even as a stochastic processing of a set of PVMs. (Recall that sharp measurements in quantum theory are represented by PVMs.) Suppose that the five-outcome POVM $\{E_0, \ldots, E_4\}$ *could* be obtained by stochastic processing of a set $\{\mathcal{P}^{(\alpha)}\}_\alpha$ of distinct binary-outcome qubit PVMs, $\mathcal{P}^{(\alpha)} \coloneqq \{\Pi_0^{(\alpha)}, \Pi_1^{(\alpha)}\}$. Specifically, this would mean that $E_y = \sum_{j,k} P(y|j,k)(\sum_\alpha \Pi_j^{(\alpha)} P(\alpha,k))$ for some auxiliary variable k, distribution $P(\alpha,k)$, and conditional $P(y|j,k)$. However, because each effect E_y is itself rank-1 and because all the $\Pi_j^{(\alpha)}$ are distinct, these sums [which are non-negative by virtue of the weights $P(y|j,k)P(\alpha,k)$ being non-negative] can each contain only a single rank-1 projector. Moreover, because postselection is not allowed in such a processing, each projector must appear in the positive sum yielding *some* effect. It follows that there must be a one-to-one association between the five effects and the full set of projectors. But this yields a contradiction. One way to see this is that no two effects are orthogonal, while the complementary rank-1 projectors appearing in a single PVM, $\Pi_0^{(\alpha)}$ and $\Pi_1^{(\alpha)}$, *are* orthogonal. (Alternatively, one can note simply that there is an odd number of effects in the POVM and an even number of projectors in the set of PVMs.)

Related work.—Reference [43] has also noted that there is an opportunity for leveraging operational equivalences among the elements of a single unsharp measurement in proofs of generalized contextuality [44]—indeed, the construction they consider is an instance of what we here termed flag convexification. The analysis in Ref. [43], however, makes use of an auxiliary assumption, namely, that the probability distribution over ontic states after a measurement is proportional to the response function of the effect that was observed. This assumption was not shown to follow from noncontextuality, so Ref. [43] does not establish our result [45].

It is also worth drawing the parallel between our Letter and that of Fritz [46,47], which established that there are Bell-like causal compatibility inequalities [46,48–50] admitting of quantum violations even in scenarios where each party implements only a single measurement, implying that incompatibility is also not necessary for exhibiting quantumness in such cases.

Conclusions.—We have demonstrated that incompatibility is not required for proofs of generalized contextuality. This fact shows a new sense in which the failure of generalized noncontextuality can be proven in a broader range of scenarios than the failure of Kochen-Specker noncontextuality. This result also opens the door to the possibility of finding new quantum advantages for information processing by considering operational scenarios with no incompatibility, some of which might have previously been overlooked by virtue of mistakenly being thought to offer no opportunity for nonclassical phenomena.

Our companion paper, Ref. [4], generalizes the results in this paper in a number of ways. First, the arguments therein are formulated in the context of arbitrary generalized probabilistic theories rather than just quantum theory. This means that if a given generalized probabilistic theory admits of a proof of contextuality in a prepare-measure scenario, then one can find such a proof involving only a single measurement. Second, we demonstrate the possibility of proofs of contextuality (for both quantum theory and generalized probabilistic theories) in scenarios that have only a single preparation device as well as a single measurement device. Such scenarios have no external inputs at all. Hence, in addition to not requiring any incompatibility, they also do not require the freedom of choice assumption—roughly, that one can choose one's external inputs freely. Finally, we show that if one's detectors are inefficient, in that they have probability p of performing ideally and probability $1-p$ of failing to give any outcome, then one can still witness contextuality, even for *arbitrary* inefficiencies, i.e., any $p > 0$. That is, there is no detector loophole for tests of generalized noncontextuality.

This research was supported by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science. D. S. and A. B. S. acknowledge support by the Foundation for Polish Science (IRAP project, ICTQT, Contract No. 2018/MAB/5, cofinanced by EU within Smart Growth Operational Programme). J. H. S. was supported by the National Science Centre, Poland (Opus project, Categorical Foundations of the Non-Classicality of Nature, Project No. 2021/41/B/ST2/03149). R. K. is supported by the Chargé de Recherche fellowship of the Fonds de la Recherche Scientifique—FNRS (F.R.S.-FNRS), Belgium. All of the diagrams within this manuscript were prepared using TikZit. Figures were prepared using Tikz.

*john.h.selby@gmail.com

[1] R. W. Spekkens, Contextuality for preparations, transformations, and unsharp measurements, Phys. Rev. A **71**, 052108 (2005).

[2] D. Schmid, J. H. Selby, and R. W. Spekkens, Unscrambling the omelette of causation and inference: The framework of causal-inferential theories, arXiv:2009.03297.

[3] R. W. Spekkens, Quasi-quantization: Classical statistical theories with an epistemic restriction, in *Quantum Theory: Informational Foundations and Foils*, edited by G. Chiribella and R. W. Spekkens (Springer, Netherlands, Dordrecht, 2016), pp. 83–135.

[4] J. H. Selby, D. Schmid, E. Wolfe, A. B. Sainz, R. Kunjwal, and R. W. Spekkens, companion paper, Accessible

fragments of generalized probabilistic theories, cone equivalence, and applications to witnessing nonclassicality, Phys. Rev. A **107**, 062203 (2023).
[5] R. W. Spekkens, The ontological identity of empirical indiscernibles: Leibniz's methodological principle and its significance in the work of Einstein, arXiv:1909.04628.
[6] R. W. Spekkens, Negativity and Contextuality are Equivalent Notions of Nonclassicality, Phys. Rev. Lett. **101**, 020401 (2008).
[7] D. Schmid, J. H. Selby, E. Wolfe, R. Kunjwal, and R. W. Spekkens, Characterization of noncontextuality in the framework of generalized probabilistic theories, PRX Quantum **2**, 010331 (2021).
[8] F. Shahandeh, Contextuality of general probabilistic theories, PRX Quantum **2**, 010330 (2021).
[9] D. Schmid, J. H. Selby, M. F. Pusey, and R. W. Spekkens, A structure theorem for generalized-noncontextual ontological models, arXiv:2005.07161.
[10] R. D. Baldijao, R. Wagner, C. Duarte, B. Amaral, and M. Terra Cunha, Noncontextuality as a meaning of classicality in Quantum Darwinism, PRX Quantum **2**, 030351 (2021).
[11] R. W. Spekkens, D. H. Buzacott, A. J. Keehn, B. Toner, and G. J. Pryde, Preparation Contextuality Powers Parity-Oblivious Multiplexing, Phys. Rev. Lett. **102**, 010401 (2009).
[12] A. Chailloux, I. Kerenidis, S. Kundu, and J. Sikora, Optimal bounds for parity-oblivious random access codes, New J. Phys. **18**, 045003 (2016).
[13] A. Ambainis, M. Banik, A. Chaturvedi, D. Kravchenko, and A. Rai, Parity oblivious d-level random access codes and class of noncontextuality inequalities, Quantum Inf. Process. **18**, 111 (2019).
[14] D. Saha, P. Horodecki, and M. Pawłowski, State independent contextuality advances one-way communication, New J. Phys. **21**, 093057 (2019).
[15] R. Raussendorf, Contextuality in measurement-based quantum computation, Phys. Rev. A **88**, 022322 (2013).
[16] M. J. Hoban, E. T. Campbell, K. Loukopoulos, and D. E. Browne, Non-adaptive measurement-based quantum computation and multi-party Bell inequalities, New J. Phys. **13**, 023014 (2011).
[17] D. Schmid, H. Du, J. H. Selby, and M. F. Pusey, The Stabilizer Subtheory has a Unique Noncontextual Model, Phys. Rev. Lett. **129**, 120403 (2022).
[18] D. Schmid and R. W. Spekkens, Contextual Advantage for State Discrimination, Phys. Rev. X **8**, 011015 (2018).
[19] M. Lostaglio and G. Senno, Contextual advantage for state-dependent cloning, Quantum **4**, 258 (2020).
[20] M. Lostaglio, Certifying Quantum Signatures in Thermodynamics and Metrology via Contextuality of Quantum Linear Response, Phys. Rev. Lett. **125**, 230603 (2020).
[21] M. F. Pusey, Anomalous Weak Values Are Proofs of Contextuality, Phys. Rev. Lett. **113**, 200401 (2014).
[22] R. Kunjwal, M. Lostaglio, and M. F. Pusey, Anomalous weak values and contextuality: Robustness, tightness, and imaginary parts, Phys. Rev. A **100**, 042116 (2019).
[23] S. A. Yadavalli and R. Kunjwal, Contextuality in entanglement-assisted one-shot classical communication, Quantum **6**, 839 (2022).
[24] R. W. Spekkens, The status of determinism in proofs of the impossibility of a noncontextual model of quantum theory, Found. Phys. **44**, 1125 (2014).
[25] A. Tavakoli and R. Uola, Measurement incompatibility and steering are necessary and sufficient for operational contextuality, Phys. Rev. Res. **2**, 013011 (2020).
[26] D. Gottesman, Stabilizer codes and quantum error correction, Ph.D. thesis, California Institute of Technology, 1997.
[27] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).
[28] D. Schmid, R. W. Spekkens, and E. Wolfe, All the noncontextuality inequalities for arbitrary prepare-and-measure experiments with respect to any fixed set of operational equivalences, Phys. Rev. A **97**, 062103 (2018).
[29] R. Kunjwal and R. W. Spekkens, From the Kochen-Specker Theorem to Noncontextuality Inequalities without Assuming Determinism, Phys. Rev. Lett. **115**, 110403 (2015).
[30] M. D. Mazurek, M. F. Pusey, R. Kunjwal, K. J. Resch, and R. W. Spekkens, An experimental test of noncontextuality without unphysical idealizations, Nat. Commun. **7** (2016).
[31] R. Kunjwal, Contextuality beyond the Kochen-Specker theorem, arXiv:1612.07250.
[32] R. Kunjwal and R. W. Spekkens, From statistical proofs of the Kochen-Specker theorem to noise-robust noncontextuality inequalities, Phys. Rev. A **97**, 052110 (2018).
[33] R. Kunjwal, Beyond the Cabello-Severini-Winter framework: Making sense of contextuality without sharpness of measurements, Quantum **3**, 184 (2019).
[34] R. Kunjwal, Hypergraph framework for irreducible noncontextuality inequalities from logical proofs of the Kochen-Specker theorem, Quantum **4**, 219 (2020).
[35] P. Busch, M. Grabowski, and P. J. Lahti, *Operational Quantum Physics* (Springer Science & Business Media, New York, 1997), Vol. 31.
[36] Y.-C. Liang, R. W. Spekkens, and H. M. Wiseman, Specker's parable of the overprotective seer: A road to contextuality, nonlocality and complementarity, Phys. Rep. **506**, 1 (2011).
[37] L. Guerini, J. Bavaresco, M. Terra Cunha, and A. Acín, Operational framework for quantum measurement simulability, J. Math. Phys. (N.Y.) **58**, 092102 (2017).
[38] An equivalent definition of compatibility can be obtained by replacing stochastic postprocessing by coarse graining (see, e.g., Lemma 1 of Ref. [37]). For sharp measurements, it is well known that this definition of compatibility reduces to commutation of the associated Hermitian operators (see Proposition 8 of Ref. [39] and Theorem 1 of Ref. [40] for proofs of some generalizations of this result).
[39] T. Heinosaari, D. Reitzner, and P. Stano, Notes on joint measurability of quantum observables, Found. Phys. **38**, 1133 (2008).
[40] R. Kunjwal, A note on the joint measurability of POVMs and its implications for contextuality, arXiv:1403.0470.
[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.130.230201 for the proof of Theorem 1 and for a more detailed analysis of our final example (illustrated in Fig. 2).

[42] A. Krishna, R. W. Spekkens, and E. Wolfe, Deriving robust noncontextuality inequalities from algebraic proofs of the Kochen-Specker theorem: The Peres-Mermin square, New J. Phys. **19**, 123031 (2017).

[43] J. Singh and Arvind, Revealing quantum contextuality using a single measurement device by generalizing measurement non-contextuality, Phys. Rev. A **107**, 012201 (2023).

[44] In fact, Ref. [43] claims that the notion of noncontextuality proposed in Ref. [1] must be supplemented with an *additional* constraint: that if two effects E and E' satisfy $E' = sE$ where $s \in [0, 1]$, then the conditional probability distributions that represent them, $P(E|\lambda)$ and $P(E'|\lambda)$, must satisfy $P(E'|\lambda) = sP(E|\lambda)$. However, this constraint is not an innovation to the notion of noncontextuality, but rather is known to follow from the representation of postprocessing within any ontological model, as noted in Ref. [24] (see the proof of condition NC3 therein).

[45] Additionally, we were not able to show that this assumption follows from noncontextuality. Furthermore, if one reconceptualizes their scenario as a prepare-measure experiment, so that one can apply the techniques of Ref. [28], we do not obtain the inequalities from Ref. [43] as valid noncontextuality inequalities.

[46] T. Fritz, Beyond Bell's theorem: Correlation scenarios, New J. Phys. **14**, 103001 (2012).

[47] T. Fritz, Beyond Bell's theorem II: Scenarios with arbitrary causal structure, Commun. Math. Phys. **341**, 391 (2016).

[48] C. J. Wood and R. W. Spekkens, The lesson of causal discovery algorithms for quantum correlations: Causal explanations of Bell-inequality violations require fine-tuning, New J. Phys. **17**, 033002 (2015).

[49] E. Wolfe, R. W. Spekkens, and T. Fritz, The inflation technique for causal inference with latent variables, J. Causal Infer. **7**, 20170020 (2019).

[50] A. Tavakoli, A. Pozas-Kerstjens, M.-X. Luo, and M.-O. Renou, Bell nonlocality in networks, Rep. Prog. Phys. **85**, 056001 (2022).